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DOMAIN OF INDIFFERENCE IN SINGLE-TYPE DIFFERENTIAL GAMES OF RETENTION IN A BOUNDED TIME INTERVAL†

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Single-type games are considered in which the players' vectorgrams are described by the same symmetric convex compact set, which expands homothetically at each instant of time. The payoff [1] is the maximum value of the Minkowski function of the compact set for motion over a given time interval.

In a previous study [2], the value of a game with simple motions, in which the payoff is the minimum of a certain convex function over the motion, was calculated. There is an example [3] of a single-type game in which, up to a certain time, player I can select any admissible value of his control. The value of a game which remains constant in a certain region of the position space has been calculated for certain classes of games [4].

1. Let E be a linear space and p a fixed instant of time such that for all $t \leqslant p$ non-negative scalar functions $a(t)$ and $b(t)$ are defined which are summable over every finite interval.

Consider the differential game

$$z^{\cdot} = -a(t)u + b(t)\upsilon; \quad z, u, \upsilon \in E; \quad \lambda(u) \leqslant 1, \ \ \lambda(\upsilon) \leqslant 1 \tag{1.1}$$

The non-zero function λ: $E \to R$ is assumed to satisfy the following conditions for a Minkowski function

$$\lambda(\varepsilon z) = |\varepsilon|\lambda(z), \quad \forall \varepsilon \in R, \quad \forall z \in E; \quad 0 \leqslant \lambda(z) < +\infty$$
$$\lambda(z) - \lambda(x) \leqslant \lambda(z + x) \leqslant \lambda(z) + \lambda(x), \quad \forall x, z \in E \tag{1.2}$$

Our definitions of players' strategies and motion will follow the procedure employed in [5]. Let us consider arbitrary controls $u(t, z)$ and $\upsilon(t, z)$ for the players, subject to the constraints

$$\lambda(u(t, z)) \leqslant 1, \quad \lambda(\upsilon(t, z)) \leqslant 1 \tag{1.3}$$

Fixing an initial time $t_0 \leqslant p$, consider a partition

$$t_0 < t_1 < \ldots < t_k < t_{k+1} = p \tag{1.4}$$

Starting from the state $z(t_0) = z_0$, construct a polygonal line

$$z(t) = z(t_i) - \left(\int_{t_i}^{t} a(r)dr\right)u\big(t_i, z(t_i)\big) + \left(\int_{t_i}^{t} b(r)dr\right)\upsilon(t_i, z(t_i)), \quad t_i \leqslant t \leqslant t_{i+1} \tag{1.5}$$

It follows from (1.2) and (1.3) that

$$\big|\lambda(z(t)) - \lambda(z(\tau))\big| \leqslant \int_{\tau}^{t} (a(r) + b(r))dr, \quad \tau \leqslant t \leqslant p$$

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It follows from this inequality that the family of functions $\lambda(z(t))$ is uniformly bounded and equicontinuous in the interval $[t_0, p]$. By Arzela's theorem [6] we can extract from any sequence $\lambda(z_n(t))$ a subsequence that is uniformly convergent in $[t_0, p]$.

By a realization of the values of λ for the motion generated by controls (1.3) we mean any function $\lambda_*(t) \geqslant 0$ which is the uniform limit in $[t_0, p]$ of a sequence of functions $\lambda(z_n(t))$, where $z_n(t)$ is a sequence of polygonal lines (1.5) with partitions whose diameters tend to zero

$$m = \max_{0 \leqslant i \leqslant k} (t_{i+1} - t_i) \to 0 \tag{1.6}$$

The aim of player I is to minimize the quantity

$$\max_{t_0 \leqslant t \leqslant p} \max(\lambda_*(t) + f_1(t); f_2(t)) \tag{1.7}$$

where $f_i(t)$ are continuous functions for $t \leqslant p$. Player II tries to maximize this quantity.

Let

$$f(t) = \max_{t \leqslant \tau \leqslant p} \left(\int_t^\tau (b(r) - a(r))dr + f_1(\tau) \right) \tag{1.8}$$

$$F(t) = \max_{t \leqslant \tau \leqslant p} \max\big(f(\tau); f_2(\tau)\big) \tag{1.9}$$

Define sets in t, z space, as follows:

$$A = \{(t,z): \lambda(z) + f(t) > F(t)\}$$

$$A_0 = \{(t,z): \lambda(z) + f(t) = F(t)\}, \quad A_1 = \{(t,z): \lambda(z) + f(t) < F(t)\} \tag{1.10}$$

Define the function

$$D(t,z) = \max\{F(t);\ \lambda(z) + f(t)\} \tag{1.11}$$

Consider the following control for player I

$$u(t,z) = z / \lambda(z),\ (t,z) \in A;\ u(t,z) = u,\ \forall u: \lambda(u) \leqslant 1,\ (t,z) \overline{\in} A \tag{1.12}$$

Fix an arbitrary control (1.3) for player II, an initial state $z_0 = z(t_0)$ and any realization $\lambda_*(t)$ generated by the control (1.12).

Theorem 1.1. Under the control (1.12) it is true that

$$\max\big(\lambda_*(t) + f_1(t); f_2(t)\big) \leqslant D(t_0, z_0),\quad \forall t \in [t_0, p] \tag{1.13}$$

Proof. Denote

$$\varphi(t) = \max\{F(t); \lambda_*(t) + f(t)\} \tag{1.14}$$

By (1.11) and (1.14), $\varphi(t_0) = D$, where we have put $D = D(t_0, z_0)$.

It follows from (1.8) and (1.9) that $f_1(t) \leqslant f(t)$ and $f_2(t) \leqslant F(t)$. Therefore, if $\varphi(t) \leqslant D$ for all $t \in [t_0, p]$, then condition (1.13) will hold.

Let $\varphi(t^*) > D$ for some $t^* \in (t_0, p]$. Let $t_* = \sup\{t \in [t_0, t^*): \varphi(t) = D\}$. Then

$$\varphi(t) > D,\ \forall t \in (t_*, t^*];\ \varphi(t_*) = D \tag{1.15}$$

Hence, by (1.11), it follows that $\varphi(t) > D \geqslant F(t_0)$ for all $t \in (t^*, t^*]$. Thus, taking the monotonicity of F into account, we obtain $\varphi(t) > F(t)$. Hence, using (1.14), we obtain

$$\lambda_*(t)+f(t)>F(t),\ \ \forall t\in(t_*,t^*] \tag{1.16}$$

Let $z_n(t)$ be a sequence of polygonal lines over partitions with diameters $m_n \to 0$ such that $\lambda(z_n(t)) \to \lambda_*(t)$ uniformly in $[t_0, p]$.

Fix a number $\tau \in (t_*, p)$. Then, using inequality (1.16), we can find a number n_* such that for all $n > n_*$

$$\lambda(z_n(t))+f(t)>F(t),\ \ \forall t\in[\tau,t^*] \tag{1.17}$$

$$\lambda(z_n(t))-\int_t^r a(s)ds>0,\ \ \tau\leqslant t<r\leqslant t+m_n,\ \ r\leqslant t^* \tag{1.18}$$

Fix some polygonal line with $n > n_*$. Let $t_{i-1} < \tau \leqslant t < \ldots < t_l < t^* \leqslant t_{l+1}$ be part of its partition points.

It follows from (1.17) that the control (1.12) is equal to $u(t_j, z_n(t_j)) = (z_n(t_j))/\lambda(z_n(t_j)), i \leqslant j \leqslant l$.

Substituting this control into (1.5) and using properties (1.2), we get

$$\lambda\left(z_n(t_{j+1})\right)\leqslant\left|\lambda\left(z_n(t_j)\right)-\int_{t_j}^{t_{j+1}} a(s)ds\right|+\int_{t_j}^{t_{j+1}} b(s)ds$$

Hence, by (1.18), it follows that

$$\lambda(z_n(t_l))\leqslant\lambda(z_n(t_i))+\int_{t_i}^{t_l}\left(b(s)-a(s)\right)ds$$

Let $n \to \infty$ in this inequality, taking into account that $t_i \to \tau$, $t_l \to t^*$. Then let τ approach t_*. The result is

$$\lambda_*(t^*)\leqslant\lambda_*(t_*)+\int_{t_*}^{t^*}\left(b(s)-a(s)\right)ds$$

Hence, using the definition of the function (1.8), we obtain the inequality

$$\lambda_*(t^*)+f(t^*)\leqslant\lambda_*(t_*)+f(t_*) \tag{1.19}$$

It follows from (1.16) and (1.14) that the left-hand side of inequality (1.19) is $\varphi(t^*)$. From the second equality of (1.15) we deduce that the right-hand side of the same inequality is D. This contradicts the first inequality of (1.15).

2. Consider the game from player II's position. Fix the following control for player II

$$\begin{aligned}&\upsilon(t,z)=z/\lambda(z);\ (t,z)\in A\cup A_0,\ \lambda(z)\neq 0\\&\upsilon(t,z)=\upsilon,\ \forall\upsilon:\lambda(\upsilon)=1;\ (t,z)\in A_0,\ \lambda(z)=0\\&\upsilon(t,z)=\upsilon,\ \forall\upsilon:\lambda(\upsilon)\leqslant 1;\ (t,z)\in A_1\end{aligned} \tag{2.1}$$

Fix any control (1.3) for player I and consider an arbitrary polygonal line (1.5).

Lemma 2.1. Suppose that for some $0 \leqslant j \leqslant k$

$$(t_j,z(t_j))\in A\cup A_0 \tag{2.2}$$

Then either

$$(t_{j+1}, z(t_{j+1})) \in A \cup A_0, \quad D(t_{j+1}, z(t_{j+1})) \geqslant D(t_j, z(t_j)) \tag{2.3}$$

or there is some $t \in [t_j, t_{j+1}]$ for which

$$\lambda(z(t)) + f_1(t) \geqslant D(t_j, z(t_j)) \tag{2.4}$$

Proof. It follows from the inclusion (2.2) and formulae (1.10) and (1.11) that

$$\lambda(z(t_j)) + f(t_j) = D(t_j, z(t_j)) \tag{2.5}$$

Substitute the control (2.1) into (1.5). Using properties (1.2), we see that for all $t \in [t_j, t_{j+1}]$

$$\lambda(z(t)) \geqslant \lambda(z(t_j)) + \int_{t_j}^{t} (b(r) - a(r))dr \tag{2.6}$$

It follows from (1.8) that for some $\tau \in [t_j, p]$

$$f(t_j) = \int_{t_j}^{\tau} (b(r) - a(r))dr + f_1(\tau) \tag{2.7}$$

Let $t_j \leqslant \tau \leqslant t_{j+1}$ and put $t = \tau$. Inequality (2.4) then follows from (2.7), (2.6) and (2.5).
Let $\tau > t_{j+1}$. It then follows from (2.7) and (1.8) that

$$f(t_j) = \int_{t_j}^{t_{j+1}} (b(r) - a(r))dr + f(t_{j+1})$$

Hence, by (2.5) and (2.6), we obtain

$$\lambda(z(t_{j+1})) + f(t_{j+1}) \geqslant D(t_j, z(t_j)) \tag{2.8}$$

We infer from (1.11) and (1.9) that $D(t_j, z(t_j)) \geqslant F(t_j) \geqslant F(t_{j+1})$. Consequently, by (1.11), the left-hand side of inequality (2.8) is equal to $D(t_{j+1}, z(t_{j+1}))$ and is not less than $F(t_j+1)$. The required relationships (2.3) now follow.

Fix an arbitrary control (1.3) for player I and an initial state $z(t_0) = z_0$.

Theorem 2.1. Given the control (2.1), a realization $\lambda_*(t)$ exists which satisfies the inequality

$$\max_{t_0 \leqslant t \leqslant p} \max\big(\lambda_*(t) + f_1(t); f_2(t)\big) \geqslant D(t_0, z_0) \tag{2.9}$$

Proof. Take an arbitrary polygonal line $z(t)$ as in (1.5).
Let $(t_0, z_0) \in A \cup A_0$. Then it follows from Lemma 2.1 that for some $\tau \in [t_0, p]$

$$\max\big(\lambda(z(\tau)) + f_1(\tau); f_2(\tau)\big) \geqslant D(t_0, z_0) \tag{2.10}$$

Passing to the limit in (2.10) along polygonal lines, we deduce that inequality (2.9) holds for any realization.

Let $(t_0, z_0) \in A_1$. Then we deduce from formulae (1.10) and (1.11) that $D(t_0, z_0) = F(t_0) > \lambda(z_0) + f(t_0) \geqslant f(t_0)$. Put

$$t_* = \sup\big\{t \in [t_0, p]: F(t) = F(t_0)\big\} \tag{2.11}$$

Then

$$F(t_*) = F(t_0) = D(t_0, z_0) \tag{2.12}$$

It follows from (1.8) and (1.9) that $F(p) = \max(f_1(p); f_2(p))$. Therefore, if $t_* = p$, we infer from (2.12) that inequality (2.10) is true for $\tau = p$.

Let $t_* < p$. We can then show, via the definitions of the function (1.9) and the number (2.11), that a sequence of points r_n exists such that

$$r_n \to t_*,\ \ r_n > t_*,\ \ F(r_n) = \max(f(r_n);\ f_2(r_n)) \tag{2.13}$$

If $F(r_n) = f_2(r_n)$ for an indefinite number of terms of the sequence, then, since the functions are continuous, we obtain $F(t_*) = f_2(t_*)$. Hence, by (2.12), it follows that inequality (2.10) is true for $\tau = t_*$.

Let $F(r_n) = f(r_n)$ for an infinite number of terms of the sequence. Take a sequence of polygonal lines $z_n(n)$ over partitions with diameters $m_n \to 0$, such that the partition (1.4) for $z_n(t)$ contains the point r_n. Then we see from (1.10) that $(r_n, z_n(r_n)) \in A \cup A_0$. As shown above, for certain $\tau_n \in [r_n, p]$ inequality (2.10) will hold for broken $z_n(t)$.

Considering now a convergent subsequence, let us assume that $\tau_n \to \tau_* \in [t_0, p]$ and $\lambda(z_n(t)) \to \lambda_*(t)$ uniformly in $[t_0, p]$. Substituting $z(t) = z_n(t)$, $\tau = \tau_n$ into (2.10) and taking to the limit, we obtain inequality (2.9).

Condition 2.1. For any number $t_* < p$ satisfying the inequality $F(t_*) > F(t)$ for $t_* < t < p$ sequences of points $t_* < s_i < l_i$, $l_i \to t_*$ exist such that

$$F(t) = \max(f(t); f_2(t)),\ \ \forall t \in [s_i, l_i]$$

Theorem 2.2. Suppose that Condition 2.1 holds. Then the control (2.1) guarantees that inequality (2.9) will hold for any realization $\lambda_*(t)$.

Proof. Let $\lambda(z_n(t))$ be polygonal lines over partitions whose diameters tend to zero, and suppose they converge uniformly to some realization. It follows from condition (2.1) that for each such line, from some index on, the points of the partition (1.4) include a point r_n at which condition (2.13) holds. Consequently, as in the previous theorem, the realization indeed satisfies inequality (2.9).

Consider the following control for player II

$$\begin{aligned} &\upsilon(t,z) = z/\lambda(z);\ \ \lambda(z) \neq 0 \\ &\upsilon(t,z) = \upsilon,\ \ \lambda(\upsilon) = 1;\ \ \lambda(z) = 0 \end{aligned} \tag{2.14}$$

Theorem 2.3. The control (2.14) guarantees that condition (2.9) will hold for any realization $\lambda_*(t)$.

Proof. Let us assume that $f_2(t) < D(t_0, z_0) = D$ for all $t_0 \leqslant t \leqslant p$.

It follows from (1.8), (1.9) and (1.11) that numbers $t_0 \leqslant \tau \leqslant s \leqslant p$ exist for which one of the following two equalities holds

$$\lambda(z_0) + \int_{t_0}^{\tau} (b(r) - a(r))dr + f_1(\tau) = D \tag{2.15}$$

$$\int_{\tau}^{s} (b(r) - a(r))dr + f_1(s) = D \tag{2.16}$$

Take any polygonal line $z(t)$ (1.5) over a partition (1.4). The control (2.14) guarantees that inequality (2.6) will hold at all points of the partition.

Suppose that inequality (2.15) holds. Then it follows from (2.6) that for any polygonal line one has $\lambda(z(\tau)) + f_1(\tau) \geqslant D$. Hence it follows that inequality (2.9) is true for any realization.

Suppose that inequality (2.16) holds. If $\tau = s$, then $f_1(\tau) = D$. Consequently, inequality (2.9) is true.

Let $\tau < s$. Then the continuity of the integral and the function f_i imply that for any number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\int_{t}^{s} (b(r) - a(r))dr + f_1(s) \geqslant D - \varepsilon,\ \ \forall t \in [\tau - \delta, \tau + \delta] \tag{2.17}$$

Take a polygonal line over a partition whose diameter is less than δ. Let $\tau \leqslant t_i \leqslant \tau + \delta$. Then it follows from (2.17) that

$$\lambda(z(t_i)) + \int_{t_i}^{s} (b(r) - a(r))dr + f_1(s) \geqslant D - \varepsilon$$

Hence, by (2.6), we see that $\lambda(z(s)) + f_1(s) \geqslant D - \varepsilon$. Thus, for any realization λ_*, the left-hand side of inequality (2.9) is at least $D - \varepsilon$ is an arbitrary number, and this implies the truth of (2.9).

Remark. It follows from Theorems 1.1 and 2.2 that the function (1.11) is the value of the game and the functions (1.12) and (2.14) are optimal controls for the players. If Condition 2.1 holds, then the function (2.1) is also an optimal control for player II. The functions (1.12) and (2.1) take arbitrary admissible values over the set A_1. In this sense, A_1 is a domain of indifference.

3. Let us consider a game problem with fixed termination time which, when formalized, reduces to a game with the criterion (1.7).

Suppose that the dynamics of the players in game (1.1) are determined for times $\tau \leqslant p$ by functions $a_1(t)$ and $b_1(t)$. This may be due to a breakdown.

Player I's aim is to minimize $\lambda_*(p)$.

Let $\lambda_*(p)$ be the value of the realization at time τ. Take τ to be the initial time in a game with fixed termination time p. It can be shown [4] that the value of this game is

$$\max(\lambda_*(\tau) + f_1(\tau); f_2(\tau)); \quad f_1(\tau) = \int_{\tau}^{p} (b_2(r) - a_2(r))dr, \quad f_2(\tau) = \max_{\tau \leqslant r \leqslant p} f_1(r) \tag{3.1}$$

The quantity (3.1) is the minimum value of $\lambda_*(p)$ guaranteed to player I, provided that player II behaves in an optimum manner.

If the choice of the time τ does not depend on player I, it may be designated as the quantity (1.7). We obtain a game with payoff (1.7).

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